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Change of variables in double integrals

Spse T is a <u>C^1-transformation</u>, whose <u>Jacobian is non-zero</u>, and that maps the region S in the uv-plane onto R in the xy-plane.

Spse f is continuous on R and R and S are Type I or type II regions.

Spse T is 1-1, except perhaps on the boundary of S.

Then, $$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

<u>Ex</u> Use the transformation $x = \frac{u}{v}$ and $y = v$ to compute $\iint_R xy\,dA$, where R is the region in the first quadrant bounded by lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$ and $xy = 3$.

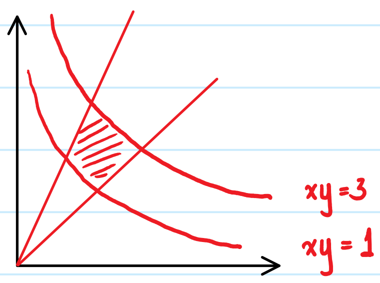
<u>Ans</u> $y = x$ is the image of $v^2 = u$

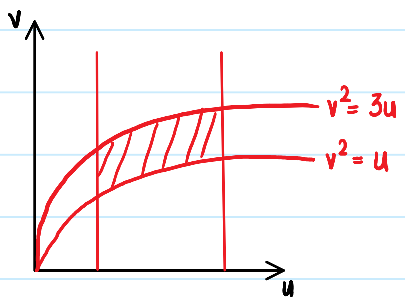
$y = 3x$ is the image of $v^2 = 3u$

Hyperbolas $xy = 1$ and $xy = 3$ are images of line $u = 1$ and $u = 3$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

Therefore $$\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right) dv\,du = \int_1^3 u\ln(\sqrt{3u} - \ln\sqrt{u})\,du = \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3}$$
$$= 2\ln 3.$$





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Spse T is a linear transformation that maps a region S in uvw-space onto a region R in the xyz-space by means of the equation $x = g(u,v,w)$, $y = h(u,v,w)$ and $z = K(u,v,w)$. Then under similar conditions (as the double integral case).

$$\boxed{\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw}$$

<u>Ex</u> Lets derive the triple integral formula for spherical coordinate

<u>Soln</u> $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

Then, $$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} + (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left[-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right] - \rho\sin\phi\left[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right]$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi = -\rho^2\sin\phi.$$

Then, as $0 \le \phi \le \pi$, $\sin\phi \ge 0$, meaning
$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = |-\rho^2\sin\phi| = \rho^2\sin\phi.$$

So finally we have,

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

<u>Ex</u> Evaluate $\iint_R \sin(9x^2+4y^2)\, dA$ where R is the region in the first quadrant bounded by the ellipse $9x^2+4y^2 = 1$.

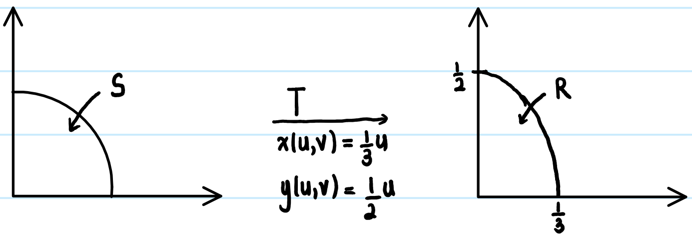
<u>Ans</u> Now we want to make the appropriate change of variables.

Let $u = 3x$ and $v = 2y$

Then $9x^2+4y^2 = u^2+v^2$

$$x = \frac{1}{3}u\,,\quad y = \frac{1}{2}v \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} = \frac{1}{6}$$

and R is the image of the quarter disc given by $u^2+v^2 \le 1$, $u \ge 0, v \ge 0$.



Then, $\displaystyle\iint_R \sin(9x^2+y^2)\, dA = \iint_S \frac{1}{6} \sin(u^2+v^2)\, du\, dv$

But now we can solve the integral over S using polar coordinates (i.e make another substitution)

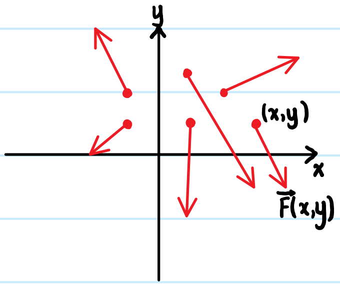
$$\iint_S \frac{1}{6}\sin(u^2+v^2)\, du\, dv = \int_0^{\pi/2}\int_0^1 \frac{1}{6}\sin(r^2)\, r\, dr\, d\theta = \frac{1}{6}\int_0^{\pi/2} d\theta \int_0^1 r\sin(r^2)\, dr = \frac{\pi}{12}\left[-\frac{1}{2}\cos(r^2)\right]_0^1 = \frac{\pi}{24}(1-\cos 1)$$

Chapter 16 Vector Calculus

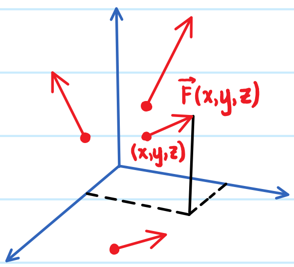
16.1 Vector fields

A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x,y) (or (x,y,z)) a two dimensional (or three dim'l) vector $\vec{F}(x,y)$ (or $\vec{F}(x,y,z)$).

• One way to picture a vector field is to draw the arrow representing the vector $\vec{F}(x,y)$ starting at the point (x,y).



Vector field on $\mathbb{R}^2$



Vector field on $\mathbb{R}^3$.

• A standard notation for vector fields $\vec{F}$ is

• $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$

Since $\vec{F}(x,y)$ is a two dim'l vector, we can write it in terms of components P and Q. sometimes called scalar fields

• $\vec{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$

Rmk $\vec{F}$ is continuous if and only if it's components P, Q and R are continuous.

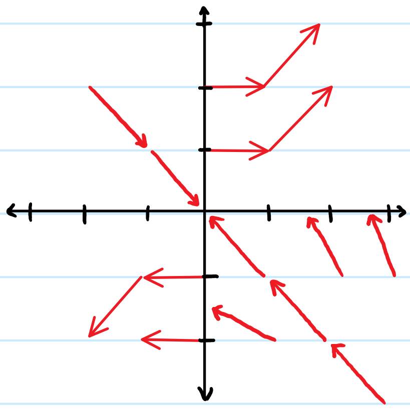
Ex 1 Sketch the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x,y) = \dfrac{y\hat{\imath} + x\hat{\jmath}}{\sqrt{x^2+y^2}}$

Ans Since $F(1,0) = \hat{\jmath}$, we draw the vector $\hat{\jmath} = \langle 0,1\rangle$ starting at the point $(1,0)$.

$F(0,1) = \hat{\imath}$, we draw the vector $\hat{\imath} = \langle 1,0\rangle$ starting at the point $(0,1)$.

Calculate $\vec{F}(x,y)$ for a view other points :

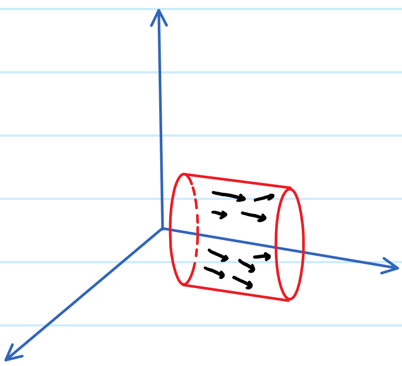
(x,y)	$\vec{F}(x,y)$	(x,y)	$\vec{F}(x,y)$
$(1,0)$	$\langle 0,1\rangle$	$(2,0)$	$\langle 0,1\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle 1,0\rangle$
$(1,1)$	$\langle \frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle$	$(-1,-1)$	$\langle -\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\rangle$



To find the rest note that $\|\vec{F}(x,y)\| = 1$.

Ex 2 Imagine a fluid flowing steadily along a pipe and let $\vec{V}(x,y,z)$ be the velocity vector at a point (x,y,z).

Then $\vec{V}$ assigns to each point (x,y,z) in a domain E, a vector in $\mathbb{R}^3$.



- The speed at any point is given by the length of the arrow.

Example 3 Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects of mass m_1 and m_2 is

$$|\vec{F}| = \frac{G m_1 m_2}{r^2}$$, where r is the distance betn the objects and G is gravitational constant

Let the object w/ mass m_1 be located at origin in $\mathbb{R}^3$.
and the position vector of the object w/ mass m_2 be $\vec{x} = \langle x, y, z \rangle$.

Then, $r = |\vec{x}|$, so $r^2 = |x|^2$.

The gravitational force exerted on this second object acts towards the origin and the unit vector in this direction is $\frac{-\vec{x}}{|\vec{x}|}$.

Thus, the gravitational force acting on the object at point $\vec{x} = \langle x, y, z \rangle$ is

$$\boxed{\vec{F}(\vec{x}) = \frac{-Gm_1m_2}{|\vec{x}|^3} \cdot \vec{x}} \longrightarrow \text{Gravitational Field.}$$

Example 4 Suppose an electric charge Q is located at the origin.

Coulumb's law : The electric force $\vec{F}(\vec{x})$ exerted by this charge q located at a point (x, y, z) is

$$\vec{F}(\vec{x}) = \frac{\varepsilon q Q}{|\vec{x}|^3} \cdot \vec{x}$$

(If q and Q are the same sign the force is repulsive
opposite " attractive)

- Electric Field : Force per unit charge.

$$\vec{E}(\vec{x}) = \frac{1}{q} \vec{F}(\vec{x}) = \frac{\varepsilon Q}{|\vec{x}|^3} \vec{x}$$

$\boxtimes$

Differentiable functions

Thm If the partial derivatives f_x, f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b). [$C^1 \Rightarrow$ differentiable].

Directional Derivatives and Gradient Vector

If $z = f(x,y)$, then the partial derivatives f_x and f_y

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} \; ; \; f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

They represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Q Want to find the rate of change of z at (x_0,y_0) in the direction of an arbitrary unit vector $\vec{u} = \langle a,b \rangle$.

- Consider the surface S w/ equation $z = f(x,y)$ and $z_0 = f(x_0,y_0)$ (the point $P(x_0,y_0,z_0)$ lies on S).

- Consider a vertical plane that passes through P in the direction of $\vec{u}$ intersects the surface S along a curve C. (Compare w/ interpretation of partial derivatives).

Then slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\vec{u}$.

DEFN The directional derivative of f at (x_0,y_0) in the direction of a unit vector $\vec{u} = \langle a,b \rangle$ is

$$D_u f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$ provided the limit exists.

Rmk $D_{\hat{\imath}} f = f_x$ and $D_{\hat{\jmath}} f = f_y$.

To compute directional derivative, we use the following theorem:

Thm If f is a differentiable function of x and y, then f has a directional derivative in the direction of any <u>unit vector</u> $\vec{u} = \langle a,b \rangle$ and

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$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$
$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$$
$$= \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{\text{Gradient of } f.} \cdot \vec{u}$$

<u>DEF</u> If f is a function of two variables x and y, then the gradient of f (grad f, ∇f) is the vector function defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial z}\hat{j}$$

<u>Ex</u> Find the directional derivative of $f(x,y) = 1 + 2x\sqrt{y}$ at the point $(3,4)$ in the direction of vector $\vec{v} = \langle 4, -3 \rangle$.

<u>Ans</u> $\nabla f(x,y) = \langle 2\sqrt{y}, \frac{x}{\sqrt{y}} \rangle$

$\nabla f(3,4) = \langle 4, 3/2 \rangle$

$\vec{v}$ is not a unit vector, but a unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{4^2+(-3)^2}} \cdot \langle 4,-3 \rangle = \langle \frac{4}{5}, \frac{-3}{5} \rangle$$

Then,

$$D_u f(3,4) = \nabla f(3,4) \cdot \vec{u} = \langle 4, \frac{3}{2} \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle = \frac{23}{10}$$

For a function of three variable,

$D_u f(x,y,z) = \nabla f \cdot \vec{u}$ where

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

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<u>Thm</u> Suppose f is a differentiable function of 2 or 3 variables. The maximum value of the directional derivative $D_u f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same dirn as the gradient vector $\nabla f(\vec{x})$.

<u>RmK</u> ∇f is a vector field on $\mathbb{R}^2$ ($\mathbb{R}^3$) and is called a gradient vector field.

<u>Ex</u> Find the gradient vector field ∇f of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

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14.5 The Chain Rule

If $y = f(x)$ and $x = g(t)$, f and g differentiable functions, then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

For a function of more than 1-variable, there are a few versions, depending on the type of compositions.

Chain Rules:

Case 1 Spse $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$, $y = h(t)$ are both differentiable functions of t.

Then z is a differentiable function of t and

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}} \quad \text{or} \quad \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Case 2 : Spse that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$. Then z is a differentiable function of s and t, and

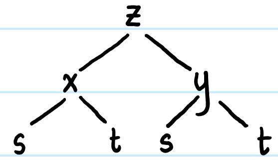
$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \;;\; \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}}$$

Case 3 Spse that u is a differentiable function of n variables $x_1, \dots, x_n$ and in turn each x_j is a a differentiable function of the m variables $t_1, \dots, t_m$.

Then u is a differentiable function of $t_1, \dots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i} \quad \text{for each } i = 1, 2, \dots, m.$$

<u>Ex</u> Use Chain rule to find $\frac{\partial z}{\partial s}$ where $z = \arcsin(x-y)$, $x = s^2$, $y = t^2 + st$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \cdot 1 = \frac{1}{\sqrt{1-(x-y)^2}} \quad , \quad \frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}} \cdot -1 = \frac{-1}{\sqrt{1-(x-y)^2}}$$

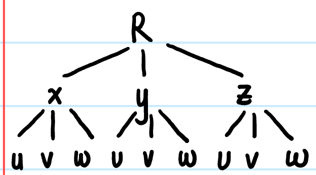
$$\frac{\partial x}{\partial s} = 2s , \quad \frac{\partial y}{\partial s} = t$$

Then,

$$\frac{\partial z}{\partial s} = \frac{1}{\sqrt{1-(x-y)^2}} \cdot 2s + \frac{-1}{\sqrt{1-(x-y)^2}} \cdot t$$

$$= \frac{2s}{\sqrt{1-(s^2+t^2+st)^2}} - \frac{t}{\sqrt{1-(s^2+t^2+st)^2}} = \frac{2s-t}{\sqrt{1-(s^2+t^2+st)^2}}$$

<u>Ex</u> Let $R(x,y,z) = x^2+y^2+z^2$, where $x = uv+vw$, $y = v^2$, $z = w^2$. Find $\frac{\partial R}{\partial w}$



$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial R}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial R}{\partial z} \cdot \frac{\partial z}{\partial w}$$

$$= 2x \cdot v + 2y \cdot 0 + 2z \cdot 2w$$

$$= 2(uv+vw)v + 4w^2 \cdot w = 2uv^2 + 2v^2w + 4w^3$$

<u>Higher Order Derivatives</u>

- For a function of two variables f, it's partial derivatives f_x, f_y are also functions of two variables, so we can consider their partial derivatives, $(f_x)_x, \ldots$, and are called the second partial derivatives.

If $z = f(x,y)$, $(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = f_{12}$

<u>Ex</u> $f(x,y) = x^2 + 2xy + y^3 x$. Compute it's second derivatives.

$f_x = 2x + 2y + y^3$; $f_y = 2x + 3y^2 x$
$f_{xx} = 2$; $f_{yy} = 6yx$
$f_{xy} = 2 + 3y^2$; $f_{yx} = 2 + 3y^2$

Note that $f_{xy} = f_{yx}$ in example.

Not a coincidence. (True for most functions one works with)

<u>Clairaut's Thm</u>

If f is defined on a ball containing the point (a,b).
If functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

<u>Maximum and minimum values</u>

A function of two variables has a local maximum [minimum] at (a,b)
if $f(x,y) \leq f(a,b)$ [$f(x,y) \geq f(a,b)$] for all points (x,y) in some disc centered at (a,b).

- $f(a,b)$ is called a local maximum [minimum] value.

<u>RmK</u> If the above inequalities hold for all (x,y) in the domain D of f (not just for a disc)
then f has an absolute maximum [minimum] at (a,b).

- A point (a,b) is called a <u>critical point</u> of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$
or one of these partial derivatives doesn't exist.

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<u>Thm</u> If f has a local max or min at (a,b), then (a,b) is a critical point of f.

Similar to single variable function converse is not true ($f(x) = x^3$ at $x = 0$)

<u>Ex</u> Find the critical points of $f(x,y) = y^2 - x^2$.

<u>Soln</u> $f_x = -2x$ & $f_y = 2y$

So the only critical point is $(0,0)$.

For points on the x-axis, $y = 0$, $f(x,y) = -x^2 < 0$. (if $x \neq 0$)

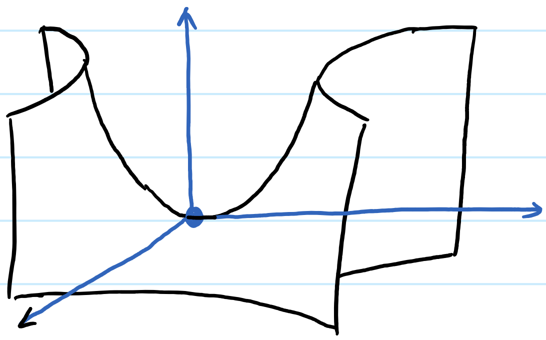
For points on the y-axis, $x = 0$, $f(x,y) = y^2 > 0$. (if $y \neq 0$)

So you can find a point near $(0,0)$ where takes smaller and larger value

Therefore, $(0,0)$ is not a local max or a local min.

However $f(0,0)$ is maximum in the dir^n of the x-axis and min in the dirn of the y-axis.

If you look at the graph near $(0,0)$ you will see



Looks like a saddle and is called the saddle point of f.

We want an algebraic way of determining whether or not a function has an extreme value at a critical point.

<u>Second Derivative Test</u> :

Spse that the second partial derivatives are continuous on a disc w/ center (a,b), and suppose $f_x(a,b)=0$ and $f_y(a,b)=0$.

Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local min.

b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local max

c) If $D<0$, then $f(a,b)$ is not a local max or min (saddle point) and graph of f crosses the tangent plane at (a,b).

<u>RmK</u> $D=0$ means test is inconclusive.